

GENERAL DECAY IN SOME TIMOSHENKO-TYPE SYSTEMS WITH THERMOELASTICITY SECOND SOUND

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ABSTRACT. In this article, we consider a vibrating nonlinear Timoshenko system with thermoelasticity with second sound. We discuss the well-posedness and the regularity of Timoshenko solution using the semi-group theory. Moreover, we establish an explicit and general decay results for a wide class of relaxing functions which depend on a stability number μ .

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1. INTRODUCTION AND SETTING OF THE PROBLEM

Beams represent the most common structural component found in civil and mechanical structures. Because of their ubiquity they are extensively studied, from an analytical viewpoint, in mechanics of materials. A widely used mathematical model for describing the transverse vibrations of beams is based on Timoshenko beam theory **TBT** (or thick beam theory) developed by Timoshenko in the 1920's. The **TBT** accounts for both the effect of rotational inertia and shear deformation that occur within a beam as it vibrates. These factors are neglected when applied to Euler-Bernoulli beam theory **EBT** (or thin beam theory), which is appropriate for beams with small cross-sectional dimensions compared to the length. In fact, a fundamental assumption in **EBT** is that cross sections remain plane and normal to the deformed longitudinal axis throughout deformation, while in **TBT** cross sections remain plane but do not remain normal to the deformed longitudinal axis as the shear deformation is taken into account. The cross section rotation from the reference to the current configuration is denoted by φ in both models. In the **EB** model, this is the same as the rotation of the longitudinal axis. In the Timoshenko model, the difference is used as measure of mean shear distortion.

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In 1921, Timoshenko [28] gave the following system of coupled hyperbolic equations

$$(1.1) \quad \begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_t - \varphi), & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

together with boundary conditions of the form

$$EI\varphi_x|_{x=0}^{x=L} = 0, \quad (u_x - \varphi)|_{x=0}^{x=L} = 0,$$

as a simple model describing the transverse vibrations of a beam. Here t denotes the time variable and x is the space variable along the beam of length L , in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

System (1.1), with the above given boundary conditions, is conservative and the natural energy of the beam, given by

$$\mathcal{E}(t) = \frac{1}{2} \int_0^L (\rho |u_t|^2 + I_\rho |\varphi_t|^2 + EI |\varphi_x|^2 + K |u_x - \varphi|^2) dx,$$

remains constant in time.

Vibration has long been known for its capacity of disturbance, discomfort, damage and destruction. Since a long time, many researchers have been investigating ways to control this phenomenon. However, with the development of control theory for partial differential equations over the last few decades, it is not surprising that the issue of stability and controllability of Timoshenko-type systems has received a great attention of many mathematicians. One effective method for vibration control is passive damping. Damping is most beneficial when used to reduce the amplitude of dynamic instabilities, or resonances, in a structure.

Damping is the conversion of mechanical energy of a structure into thermal energy. A structure subject to oscillatory deformation contains a combination of kinetic and potential energy.

A damping effect may be caused by applying the beam to internal or boundary frictional mechanisms. Depending of the nature of the beam's material, a damping effect may be rotating beam. For Viscoelastic materials with long memory, some beams are characterized by possessing both viscous and elastic behavior. As a result of this behavior, some of the energy stored in a viscoelastic system is recovered upon removal of the load, and the remainder is dissipated in the form of heat.

Kim and Renardy [7] considered (1.1) together with two boundary controls of the form

$$\begin{aligned} K\varphi(L, t) - K \frac{\partial u}{\partial x}(L, t) &= \alpha \frac{\partial u}{\partial t}(L, t) \quad \forall t \geq 0, \\ EI \frac{\partial \varphi}{\partial x}(L, t) &= -\beta \frac{\partial \varphi}{\partial t}(L, t) \quad \forall t \geq 0, \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng *et al.* [4], where the stabilization of vibrations in a Timoshenko system was studied. Raposo *et al.* [20] studied (1.1) with homogeneous Dirichlet boundary conditions and

two linear frictional dampings. Precisely, they looked into the following system

$$(1.2) \quad \begin{cases} \rho_1 u_{tt} - K(u_x - \varphi) + u_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0, & t > 0 \end{cases}$$

and proved that the energy associated with (1.2) decays exponentially. Soufyane and Wehbe [27] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. They considered

$$(1.3) \quad \begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, & \text{in } (0, L) \times \mathbb{R}_+, \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0, & t > 0, \end{cases}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. Rivera and Racke [17] obtained a similar result in a work, where the damping function $b = b(x)$ is allowed to change sign. They also in treated [16] a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0, \\ \rho_2 \psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0, \end{cases}$$

in a one-dimensional bounded domain. The dissipation here is through frictional damping which is only in the equation for the rotation angle. The authors gave an alternative proof for a sufficient and necessary condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case.

Shi and Feng [24] used the frequency multiplier method to investigate a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. The nonuniform Timoshenko beam has also been studied by Ammar-Khodja *et al.* [2] and a similar result to that in [24] has been established.

Ammar-Khodja *et al.* [1] considered a linear Timoshenko-type system with memory of the form

$$(1.4) \quad \begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x) \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \end{cases}$$

in $(0, L) \times \mathbb{R}_+$, and proved, using the multiplier techniques, that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. More precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their results. This result has been later improved by Messaoudi and Mustafa [13] and Guesmia and Messaoudi [5], where the technical

conditions on g'' have been removed and those on g' have been weakened. Also, Guesmia and Messaoudi [6] considered the following system

$$(1.5) \quad \begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - \kappa \psi_{xx} + \int_0^t g(t-\tau)(a(x)\psi_x(\tau))_x d\tau + K(\varphi_x + \psi) + b(x)h(\psi_t) = 0, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \end{cases}$$

in $(0, 1) \times \mathbb{R}_+$. They proved under similar conditions on the relaxation function g , which are similar to those in [3], and by assuming that

$$a(x) + b(x) \geq \rho > 0, \quad \forall x \in (0, 1),$$

an exponential stability for g decaying exponentially and h linear, and polynomial stability when g decays polynomially and h is nonlinear.

Concerning stabilization via heat effect, Rivera and Racke [15] investigated the following system

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 \theta_t - K\theta_{xx} + \gamma\psi_{xt} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

where φ, ψ, θ are functions of (x, t) model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions on $\sigma, \rho_i, b, K, \gamma$, they proved several exponential decay results for the linearized system and non exponential stability result for the case of different wave speeds.

Concerning Timoshenko systems of thermoelasticity with second sound, Messaoudi *et al.* [12] studied

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 \theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau_0 q_t + q + \kappa\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

where $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature difference, $q = q(x, t)$ is the heat flux vector, $\rho_1, \rho_2, \rho_3, b, k, \gamma, \delta, \kappa, \mu, \tau_0$ are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_{\psi}(0, 0) = k,$$

and

$$\sigma_{\varphi_x \varphi_x}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\psi \psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established in the presence of the extra frictional damping $\mu\varphi_t$.

Fernández Sare and Racke [3] considered

$$(1.6) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 \theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

and showed that, in the absence of the extra frictional damping ($\mu = 0$), the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of

coupling via Fourier's law [15]. This surprising property holds even for systems with history of the form

$$(1.7) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(\cdot, t-s)ds + \delta\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 \theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

Precisely, it has been shown that both systems (1.6) and (1.7) are no longer exponentially stable even for equal-wave speeds $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. However, no other rate of decay has been discussed.

Very recently, Santos et al. [22] considered (1.6) and introduced a new stability number

$$\mu = \left(\tau - \frac{\rho_1}{k\rho_3}\right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\rho_1 \delta^2 \rho_1}{kb\rho_3},$$

and used the semi-group method to obtain exponential decay result for $\mu = 0$ and a polynomial decay for $\mu \neq 0$.

The boundary feedback of memory type has also been used by Santos [21]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. This last result has been improved and generalized by Messaoudi and Soufyane [9]. For more results concerning well-posedness and controllability of Timoshenko systems, we refer the reader to [10, 11], [14], [18], [23] and [25, 26].

In this paper we consider the following Timoshenko system:

$$(1.8) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x + \alpha(t)h(\psi_t) = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\ \rho_3 \theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\ \varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = q(0, t) = q(1, t) = 0, & \forall t \geq 0, \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), & \forall x \in (0, 1), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), & \forall x \in (0, 1), \\ \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), & \forall x \in (0, 1), \end{cases}$$

where, $\rho_1, \rho_2, \rho_3, b, k, \delta, \beta$ are positive constants, $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature difference and $q = q(x, t)$ is the heat flux vector. Also, α and h are two functions to be fixed later.

Using (1.8)₁, (1.8)₃ and the boundary conditions (1.8)₅, we have

$$\frac{d^2}{dt^2} \int_0^1 \varphi(x, t) dx = 0 \text{ and } \frac{d}{dt} \int_0^1 \theta(x, t) dx = 0.$$

Consequently, we obtain

$$\int_0^1 \varphi(x, t) dx = \left(\int_0^1 \varphi_1(x) dx \right) t + \int_0^1 \varphi_0(x) dx \text{ and } \int_0^1 \theta(x, t) dx = \int_0^1 \theta_0(x) dx.$$

If we set

$$\bar{\varphi}(x, t) = \varphi(x, t) - \left(\left(\int_0^1 \varphi_1(x) dx \right) t + \int_0^1 \varphi_0(x) dx \right),$$

and

$$\bar{\theta}(x, t) = \theta(x, t) - \int_0^1 \theta_0(x) dx,$$

then $(\bar{\varphi}, \psi, \bar{\theta}, q)$ satisfy also the system (1.8), and we have

$$\int_0^1 \bar{\varphi}(x, t) dx = 0 \quad \text{and} \quad \int_0^1 \bar{\theta}(x, t) dx = 0.$$

From now on, we use the new variables $(\bar{\varphi}, \psi, \bar{\theta}, q)$, but we denote them by $(\varphi, \psi, \theta, q)$, for simplicity.

The article is organized as follows. First, in Section 2, we use the semi-group theory to prove the existence and uniqueness of solutions of system (1.8). Next, in Section 3, we study the asymptotic behavior of the energy of solutions of system (1.8) using the multiplier method. For that purpose, we assume some hypotheses on α and h . The optimal exponential and polynomial decay rate estimates can be obtained in some special cases with explicit nonlinear terms.

2. WELL-POSEDNESS AND REGULARITY

In this section, we discuss the well-posedness of the problem (1.8), using the semi-group theory. We consider the following hypotheses on α and h :

$(A_1) : \alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is differentiable and decreasing.

$(A_2) : h : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function satisfying $h(0) = 0$.

We introduce the Hilbert space:

$$L_\star^2(0, 1) = \{v \in L^2(0, 1) : \int_0^1 v(s) ds = 0\},$$

$$H_\star^1(0, 1) = H^1(0, 1) \cap L_\star^2(0, 1),$$

$$H_\star^2(0, 1) = \{v \in H^2(0, 1) : v_x(0) = v_x(1) = 0\}.$$

The energy associated with the system (1.8) is defined by:

$$E(\varphi, \psi, \theta, q)(t) = \frac{1}{2} \int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b \psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau q^2) dx.$$

Let

$$H = H_\star^1(0, 1) \times L_\star^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1) \times L_\star^2(0, 1) \times L^2(0, 1),$$

be the Hilbert space endowed with the inner product defined, for

$U = (u_1, u_2, u_3, u_4, u_5, u_6)^t \in H$, $V = (v_1, v_2, v_3, v_4, v_5, v_6)^t \in H$, by

$$\begin{aligned} \langle U, V \rangle_H &= \rho_1 \langle u_2, v_2 \rangle_{L^2(0,1)} + \rho_2 \langle u_4, v_4 \rangle_{L^2(0,1)} + k \langle u_{1x} + u_3, v_{1x} + v_3 \rangle_{L^2(0,1)} \\ &\quad + b \langle u_{3x}, v_{3x} \rangle_{L^2(0,1)} + \rho_3 \langle u_5, v_5 \rangle_{L^2(0,1)} + \tau \langle u_6, v_6 \rangle_{L^2(0,1)}. \end{aligned}$$

For $\Phi = (\varphi, u, \psi, v, \theta, q)^t$ and $\Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)^t$, where $u = \varphi_t$ and $v = \psi_t$, (1.8) is equivalent to the abstract first order Cauchy problem

$$(2.1) \quad \begin{cases} \frac{d}{dt}\Phi(t) + (A + B)\Phi(t) = 0, & \forall t \in \mathbb{R}_+, \\ \Phi(0) = \Phi_0, \end{cases}$$

where $A : D(A) \subset H \longrightarrow H$ is the linear operator defined by

$$(2.2) \quad A\Phi = \begin{pmatrix} -u \\ -\frac{k}{\rho_1}\varphi_{xx} - \frac{k}{\rho_1}\psi_x \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\delta}{\rho_2}\theta_x \\ \frac{1}{\rho_3}q_x + \frac{\delta}{\rho_3}v_x \\ \frac{\beta}{\tau}q + \frac{1}{\tau}\theta_x \end{pmatrix},$$

and $B : D(B) \subset H \longrightarrow H$ is the nonlinear operator defined by

$$B\Phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha(t)h(v) \\ 0 \\ 0 \end{pmatrix}.$$

The domain of the operator A is given by $D(A) = \{\Phi \in H ; A\Phi \in H\}$ and endowed with the graph norm

$$\|\Phi\|_{D(A)} = \|\Phi\|_H + \|A\Phi\|_H,$$

can be characterized by

$$\begin{aligned} D(A) &= (H_*^2(0, 1) \cap H_*^1(0, 1)) \times H_*^1(0, 1) \times (H^2(0, 1) \cap H_0^1(0, 1)) \\ &\quad \times H_0^1(0, 1) \times H_*^1(0, 1) \times H_0^1(0, 1). \end{aligned}$$

The domain of the operator B is given by $D(B) = \{\Phi \in H ; B\Phi \in H\} = H$.

We first state and prove the following lemmas which will be useful to deduce the well-posedness result.

Lemma 2.1. *For $\Phi \in D(A)$, we have $(A\Phi, \Phi)_H \geq 0$.*

Proof. For any $\Phi = (\varphi, u, \psi, v, \theta, q)^t \in D(A)$, we have

$$\begin{aligned} (A\Phi, \Phi)_H &= k \int_0^1 -(u_x + v)(\varphi_x + \psi) dx + \int_0^1 (-k\varphi_{xx} - k\psi_x)u dx + b \int_0^1 -v_x\psi_x dx \\ &\quad + \int_0^1 (-b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x)v dx + \int_0^1 (q_x + \delta v_x)\theta dx + \int_0^1 (\beta q + \theta_x)q dx. \end{aligned}$$

Using integration by parts and the boundary conditions in (1.8), we obtain

$$(A\Phi, \Phi)_H = \beta \int_0^1 q^2 dx \geq 0.$$

This ends the proof of the lemma. ■

Lemma 2.2. *$I + A$ is a surjective operator.*

Proof. For any $W = (w_1, w_2, w_3, w_4, w_5, w_6) \in H$, we prove that there exists $V = (v_1, v_2, v_3, v_4, v_5, v_6) \in D(A)$ satisfying

$$(I + A)V = W.$$

That is,

$$(2.3) \quad \begin{cases} -v_2 + v_1 = w_1, \\ -kv_{1xx} - kv_{3x} + \rho_1 v_1 = \rho_1(w_1 + w_2), \\ -v_4 + v_3 = w_3, \\ -bv_{3xx} + k(v_{1x} + v_3) + \delta v_{5x} + \rho_2 v_4 = \rho_2 w_4, \\ v_{6x} + \delta v_{4x} + \rho_3 v_5 = \rho_3 w_5, \\ (\beta + \tau)v_6 + v_{5x} = \tau w_6. \end{cases}$$

Then (2.3)₁, (2.3)₃ and (2.3)₅ yield

$$(2.4) \quad v_2 = v_1 - w_1 \in H_*^1(0, 1),$$

$$(2.5) \quad v_4 = v_3 - w_3 \in H_0^1(0, 1),$$

$$v_{6x} = \rho_3 w_5 + \delta w_{3x} - \delta v_{3x} - \rho_3 v_5.$$

By integration over $(0, x)$ and using $v_6(0) = w_3(0) = v_3(0) = 0$, we obtain

$$(2.6) \quad v_6 = \rho_3 \int_0^x w_5 ds + \delta w_3 - \delta v_3 - \rho_3 \int_0^x v_5 ds.$$

We substitute (2.6) into (2.3)₆ and we get

$$v_{5x} + (\beta + \tau) \left[\rho_3 \int_0^x w_5 ds + \delta w_3 - \delta v_3 - \rho_3 \int_0^x v_5 ds \right] = \tau w_6.$$

Hence, we deduce that

$$(2.7) \quad -v_{5x} + (\beta + \tau)\delta v_3 + \rho_3(\beta + \tau) \int_0^x v_5 ds = (\beta + \tau)\delta w_3 + (\beta + \tau)\rho_3 \int_0^x w_5 ds - \tau w_6.$$

Again, we substitute (2.7) into (2.3)₄, we get

$$\begin{aligned} & -bv_{3xx} + kv_{1x} + kv_3 + \delta \left[(\beta + \tau)\delta v_3 + \rho_3(\beta + \tau) \int_0^x v_5 ds - (\beta + \tau)\delta w_3 \right. \\ & \quad \left. - (\beta + \tau)\rho_3 \int_0^x w_5 ds - \tau w_6 \right] + \rho_2 v_3 = \rho_2(w_3 + w_4), \end{aligned}$$

and we infer that

$$(2.8) \quad \begin{aligned} & -bv_{3xx} + kv_{1x} + kv_3 + \delta^2(\beta + \tau)\delta v_3 + \rho_3\delta(\beta + \tau) \int_0^x v_5 ds + \rho_2 v_3 = (\beta + \tau)\delta^2 w_3 \\ & + (\beta + \tau)\delta\rho_3 \int_0^x w_5 ds - \delta\tau w_6 + \rho_2(w_3 + w_4). \end{aligned}$$

By using (2.7), (2.8) and (2.3)₂, it can be shown that v_1 , v_3 and v_5 satisfy

$$(2.9) \quad \begin{cases} -kv_{1xx} - kv_{3x} + \rho_1 v_1 = h_1 \in L_*^2(0, 1), \\ -bv_{3xx} + kv_{1x} + kv_3 + (\delta^2(\beta + \tau) + \rho_2)v_3 + \rho_3\delta(\beta + \tau) \int_0^x v_5 ds = h_2 \in L^2(0, 1), \\ -\rho_3 v_{5x} + \rho_3(\beta + \tau)\delta v_3 + \rho_3^2(\beta + \tau) \int_0^x v_5 = h_3 \in L^2(0, 1), \end{cases}$$

where

$$\begin{cases} h_1 = \rho_1(w_1 + w_2), \\ h_2 = (\beta + \tau)\delta^2 w_3 + (\beta + \tau)\delta\rho_3 \int_0^x w_5 ds - \delta\tau w_6 + \rho_2(w_3 + w_4), \\ h_3 = \rho_3(\beta + \tau)\delta w_3 + (\beta + \tau)\rho_3^2 \int_0^x w_5 ds - \rho_3\tau w_6. \end{cases}$$

Let $u = (u_1, u_3, u_5)$ and $v = (v_1, v_3, v_5)$, a simple multiplication of (2.9)₁, (2.9)₂ and (2.9)₃, by u_1, u_3 and $\int_0^x u_5 ds$ respectively, and integration over $(0, 1)$ yield

$$\begin{aligned} (2.10) \quad & \bullet \quad -k \int_0^1 v_{1xx} u_1 dx - k \int_0^1 v_{3x} u_1 dx + \rho_1 \int_0^1 v_1 u_1 dx = \int_0^1 h_1 u_1 dx, \\ & \bullet \quad -b \int_0^1 v_{3xx} u_3 dx + k \int_0^1 v_{1x} u_3 dx + k \int_0^1 v_3 u_3 dx + (\delta^2(\beta + \tau) + \rho_2) \int_0^1 v_3 u_3 dx \\ & \quad + \rho_3 \delta(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) u_3 dx = \int_0^1 h_2 u_3 dx, \\ & \bullet \quad -\rho_3 \int_0^1 v_{5x} \left(\int_0^x u_5 ds \right) dx + \rho_3(\beta + \tau)\delta \int_0^1 v_3 \left(\int_0^x u_5 ds \right) dx + \\ & \quad \rho_3^2(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) \left(\int_0^x u_5 ds \right) dx = \int_0^1 h_3 \left(\int_0^x u_5 ds \right) dx. \end{aligned}$$

Using integration by parts and the boundary conditions yield

$$\begin{aligned} & \bullet \quad k \int_0^1 v_{1x} u_{1x} dx + k \int_0^1 v_{3x} u_{1x} dx + \rho_1 \int_0^1 v_1 u_1 dx = \int_0^1 h_1 u_1 dx, \\ & \bullet \quad b \int_0^1 v_{3x} u_{3x} dx + k \int_0^1 v_{1x} u_3 dx + k \int_0^1 v_3 u_3 dx + (\delta^2(\beta + \tau) + \rho_2) \int_0^1 v_3 u_3 dx \\ & \quad + \rho_3 \delta(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) u_3 dx = \int_0^1 h_2 u_3 dx, \\ & \bullet \quad \rho_3 \int_0^1 v_{5x} u_5 dx + \rho_3(\beta + \tau)\delta \int_0^1 v_3 \left(\int_0^x u_5 ds \right) dx + \\ & \quad \rho_3^2(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) \left(\int_0^x u_5 ds \right) dx = \int_0^1 h_3 \left(\int_0^x u_5 ds \right) dx. \end{aligned}$$

The sum of the previous equations gives the following variational formulation

$$(2.11) \quad b(v, u) = l(u),$$

for all $u = (u_1, u_3, u_5) \in H_*^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)$, where b is defined by

$$\begin{aligned} b(v, u) = & k \int_0^1 (v_{1x} + v_3)(u_{1x} + u_3) dx + \rho_1 \int_0^1 v_1 u_1 dx + b \int_0^1 v_{3x} u_{3x} dx \\ & + (\delta^2(\beta + \tau) + \rho_2) \int_0^1 v_3 u_3 dx + \rho_3 \delta(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) u_3 dx + \rho_3 \int_0^1 v_{5x} u_5 dx \\ & + \rho_3(\beta + \tau)\delta \int_0^1 v_3 \left(\int_0^x u_5 ds \right) dx + \rho_3^2(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right) \left(\int_0^x u_5 ds \right) dx, \end{aligned}$$

and l is defined by

$$l(u) = \int_0^1 h_1 u_1 dx + \int_0^1 h_2 u_3 dx + \int_0^1 h_3 \left(\int_0^x u_5 ds \right) dx.$$

We introduce the Hilbert space $\Lambda = H_*^1(0, 1) \times H_0^1(0, 1) \times L^2(0, 1)$ equipped with the norm

$$\|v\|_\Lambda^2 = \|v_{1x} + v_3\|_2^2 + \|v_1\|_2^2 + \|v_{3x}\|_2^2 + \|v_5\|_2^2.$$

It is clear that b is a bilinear and continuous form on $\Lambda \times \Lambda$, and l is a linear and continuous form on Λ . Furthermore, there exists a positive constant c_0 such that

$$\begin{aligned} b(v, v) &= k\|v_{1x} + v_3\|_2^2 + \rho_1\|v_1\|_2^2 + b\|v_{3x}\|_2^2 + (\delta^2(\beta + \tau) + \rho_2)\|v_3\|_2^2 + \rho_3\|v_5\|_2^2 \\ &\quad + 2\rho_3(\beta + \tau)\delta \int_0^1 v_3 \left(\int_0^x v_5 ds \right) dx + \rho_3^2(\beta + \tau) \int_0^1 \left(\int_0^x v_5 ds \right)^2 dx \\ &\geq c_0\|v\|_\Lambda^2. \end{aligned}$$

which implies that b is coercive.

Therefore, using the Lax-Milgram theorem we conclude that the system (2.9) has a unique solution

$$(v_1, v_3, v_5) \in (H_*^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)),$$

and we deduce from (2.4)-(2.6) the existence of $v_2 \in H_*^1(0, 1)$, $v_4 \in H_0^1(0, 1)$, and $v_6 \in L_*^2(0, 1) \subset L^2(0, 1)$.

Now, it remains to show that

$$v_1 \in H_*^2(0, 1) \cap H_*^1(0, 1), \quad v_3 \in H^2(0, 1) \cap H_0^1(0, 1), \quad v_5 \in H_*^1(0, 1) \quad \text{and} \quad v_6 \in H_0^1(0, 1).$$

From (2.9), we have

$$-kv_{1xx} = kv_{3x} - \rho_1 v_1 + h_1 \in L^2(0, 1).$$

Consequently, it follows that

$$v_1 \in H^2(0, 1) \cap H_*^1(0, 1).$$

Moreover, (2.10) is also true for any $\varphi_1 \in \mathcal{C}^1([0, 1])$. Hence, we have

$$k \int_0^1 v_{1x} \varphi_{1x} dx + k \int_0^1 v_3 \varphi_{1x} dx + \rho_1 \int_0^1 v_1 \varphi_1 dx = \int_0^1 h_1 \varphi_1 dx,$$

for any $\varphi_1 \in \mathcal{C}^1([0, 1])$. Thus, using integration by parts we obtain

$$v_{1x}(1)\varphi_1(1) - v_{1x}(0)\varphi_1(0) = 0, \quad \text{for all } \varphi_1 \in \mathcal{C}^1([0, 1]).$$

Therefore, $v_{1x}(1) = v_{1x}(0) = 0$, and we deduce that

$$v_1 \in H_*^2(0, 1) \cap H_*^1(0, 1).$$

Now, we substitute (2.3)₆ into (2.3)₄, we get

$$bv_{3xx} = kv_{1x} + kv_3 + \delta\tau w_6 - \delta(\beta + \tau)v_6 + \rho_2 v_3 - h_2 \in L^2(0, 1).$$

Consequently, it follows that

$$v_3 \in H^2(0, 1) \cap H_0^1(0, 1).$$

On the other hand, we get from (2.3)₆,

$$v_{5x} = \tau w_6 - (\beta + \tau)v_6 \in L^2(0, 1),$$

and we deduce that

$$v_5 \in H^1(0, 1) \cap L_*^2(0, 1).$$

Similarly, from (2.3) we have

$$v_{6x} = \rho_3 w_5 + \delta w_{3x} - \delta v_{3x} - \rho_3 v_5 \in L^2(0, 1) \quad \text{which implies} \quad v_6 \in H_0^1(0, 1),$$

as $v_6(0) = v_6(1) = 0$.

Finally, the operator $I + A$ is surjective. ■

Using Lemmas 2.1 and 2.2, we conclude that the operator $A + B$ is the infinitesimal generator of a non-linear contraction C_0 -semi-group on the Hilbert space H . Finally, by applying the semi-group theory to (2.1) (see [8, 19]), we easily get the following well-posedness result.

Theorem 2.1. *Assume that (A_1) and (A_2) are satisfied, then for all initial data*

$$(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in (H_\star^2(0, 1) \cap H_\star^1(0, 1)) \times H_\star^1(0, 1) \times (H^2(0, 1) \cap H_0^1(0, 1)) \\ \times H_0^1(0, 1) \times H_\star^1(0, 1) \times H_0^1(0, 1),$$

the system (1.8) has a unique solution $(\varphi, \psi, \theta, q)$ that verifies

$$(\varphi, \psi) \in C^0(\mathbb{R}_+, (H_\star^2(0, 1) \cap H_\star^1(0, 1)) \times (H^2(0, 1) \cap H_0^1(0, 1))) \\ \cap C^1(\mathbb{R}_+, H_\star^1(0, 1) \times H_0^1(0, 1)) \cap C^2(\mathbb{R}_+, L_\star^2(0, 1) \times L^2(0, 1)),$$

and

$$(\theta, q) \in C^0(\mathbb{R}_+, H_\star^1(0, 1) \times H_0^1(0, 1)) \cap C^1(\mathbb{R}_+, L_\star^2(0, 1) \times L^2(0, 1)).$$

3. STABILITY RESULTS

In this section, we state and prove a stability result for the nonlinear Timoshenko system (1.8). For this purpose, we consider the following hypotheses:

$(A_1) : \alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable and decreasing function.

$(A_2)^* : h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous non-decreasing function such that $h(0) = 0$ and there exists a continuous strictly increasing odd function $h_0 \in C([0, +\infty))$, continuously differentiable in a neighborhood of 0 and satisfying $h_0(0) = 0$

$$\begin{cases} h_0(|(s)|) \leq |h(s)| \leq h_0^{-1}(|(s)|), & \text{for all } |s| \leq \varepsilon, \\ c_1|s| \leq |h(s)| \leq c_2|s|, & \text{for all } |s| \geq \varepsilon. \end{cases}$$

where $c_i > 0$ for $i = 1, 2$.

Moreover, we define a function H by

$$(3.1) \quad H(x) = \sqrt{x}h_0(\sqrt{x})$$

Thanks to Assumption $(A_2)^*$, H is of class C^1 and is strictly convex on $(0, r^2]$, where $r > 0$ is a sufficiently small number.

Remark 1.

- We denote by c positive generic constant throughout this paper.
- The hypothesis A_1 implies that $\alpha(t) \leq c$.

We recall here the stability number defined by :

$$\mu = \left[\left(\tau - \frac{\rho_1}{k\rho_3} \right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau\delta^2\rho_1}{bk\rho_3} \right].$$

3.1. The case $\mu = 0$.

In this part, we state and prove the decay results which are not necessarily of exponential or polynomial types. For this purpose, we establish several lemmas. We recall that the energy associated with the system (1.8) is defined by

$$(3.2) \quad E(t) := \frac{1}{2} \int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3\theta^2 + \tau q^2) dx.$$

Throughout the rest of this paper we assume that conditions (A_1) and $(A_2)^*$ hold.

Lemma 3.1. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional E satisfies*

$$(3.3) \quad E'(t) = -\beta \int_0^1 q^2 dx - \alpha(t) \int_0^1 \psi_t h(\psi_t) dx \leq 0.$$

Proof. By multiplying the first four equations in (1.8), respectively, by φ_t , ψ_t , θ and q , using the integration by parts with respect to x over $(0, 1)$, the boundary conditions $(1.8)_5$ and the hypotheses (A_1) and $(A_2)^*$, we obtain (3.3). ■

Lemma 3.2. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.4) \quad K_1(t) := - \int_0^1 (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx,$$

verifies the following estimate

$$(3.5) \quad \begin{aligned} K_1'(t) \leq & - \rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + c \int_0^1 \psi_x^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{\delta}{2} \int_0^1 \theta^2 dx + \frac{1}{2} \int_0^1 h^2(\psi_t) dx. \end{aligned}$$

Proof. By differentiating (3.4) and using the first and second equations of (1.8), we get

$$\begin{aligned} K_1'(t) = & - \rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx - \int_0^1 k(\varphi_x + \psi)_x \varphi dx - \int_0^1 (b\psi_{xx} - k(\varphi_x + \psi) \\ & - \delta\theta_x - \alpha(t)h(\psi_t))\psi dx. \end{aligned}$$

Integrating by parts and using the boundary conditions $(1.8)_5$, we have

$$\begin{aligned} K_1'(t) = & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + b \int_0^1 \psi_x^2 dx + \int_0^1 k(\varphi_x + \psi)^2 dx \\ & - \delta \int_0^1 \theta \psi_x dx + \int_0^1 \alpha(t) h(\psi_t) \psi dx. \end{aligned}$$

Applying Young's inequality, we obtain (3.5). ■

Lemma 3.3. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.6) \quad K_2(t) := \rho_2 \int_0^1 \psi \psi_t dx - \rho_2 \int_0^1 \varphi_t w dx - \delta \tau \int_0^1 \psi q dx,$$

satisfies, for any $\varepsilon > 0$

$$(3.7) \quad \begin{aligned} K_2'(t) \leq & - (b - 2c\varepsilon) \int_0^1 \psi_x^2 dx + c \left(\int_0^1 \psi_t^2 dx + \int_0^1 q^2 dx + \int_0^1 h^2(\psi_t) dx \right) \\ & + \rho_1 \varepsilon \int_0^1 \varphi_t^2 dx, \end{aligned}$$

where w is the solution of the problem

$$(3.8) \quad \begin{cases} -w_{xx} = \psi_x, \\ w(0) = w(1) = 0. \end{cases}$$

Proof. By differentiation of (3.6) and the use of the first, second and fourth equations of (1.8), we get

$$\begin{aligned} K'_2(t) = & \rho_2 \int_0^1 \psi_t^2 dx + b \int_0^1 \psi_{xx} \psi dx - k \int_0^1 (\varphi_x + \psi) \psi dx - \delta \int_0^1 \theta_x \psi dx - \alpha(t) \int_0^1 \psi h(\psi_t) dx \\ & + k \int_0^1 (\varphi_x + \psi)_x w dx + \rho_1 \int_0^1 \varphi_t w_t dx - \tau \delta \int_0^1 \psi_t q dx + \delta \beta \int_0^1 \psi q dx + \delta \int_0^1 \theta_x \psi dx. \end{aligned}$$

Integrating by parts the last equality, using (3.8) and the boundary conditions (1.8)₅, we have

$$\begin{aligned} K'_2(t) = & \rho_2 \int_0^1 \psi_t^2 dx - b \int_0^1 \psi_x^2 dx - k \int_0^1 \psi^2 dx + k \int_0^1 w_x^2 dx - \alpha(t) \int_0^1 \psi h(\psi_t) dx \\ & + \rho_1 \int_0^1 \varphi_t w_t dx - \tau \delta \int_0^1 \psi_t q dx + \delta \beta \int_0^1 \psi q dx. \end{aligned}$$

By a simple calculation, we easily deduce that the function w satisfies the following estimates

$$(3.9) \quad \int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx,$$

$$(3.10) \quad \int_0^1 w_t^2 dx \leq c \int_0^1 \psi_t^2 dx.$$

Thanks to Young's and Poincaré's inequalities and (3.9)-(3.10), we conclude that

$$\begin{aligned} (3.11) \quad K'_2(t) \leq & \rho_2 \int_0^1 \psi_t^2 dx - b \int_0^1 \psi_x^2 dx + \frac{\rho_1}{4\varepsilon} \int_0^1 w_t^2 dx + \rho_1 \varepsilon \int_0^1 \varphi_t^2 dx \\ & + \tau \delta \varepsilon \int_0^1 \psi_t^2 dx + \frac{\tau \delta}{4\varepsilon} \int_0^1 q^2 dx + c_p \varepsilon \int_0^1 \psi_x^2 dx + \frac{(\delta \beta)^2}{4\varepsilon} \int_0^1 q^2 dx \\ & + \varepsilon c_p \int_0^1 \psi_x^2 dx + \frac{c^2}{4\varepsilon} \int_0^1 h^2(\psi_t) dx. \end{aligned}$$

Therefore, we obtain (3.7). ■

Lemma 3.4. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.12) \quad K_3(t) := -\tau \rho_3 \int_0^1 q \left(\int_0^x \theta(t, y) dy \right) dx,$$

satisfies

$$(3.13) \quad K'_3(t) \leq -\frac{\rho_3}{2} \int_0^1 \theta^2 dx + c \left(\int_0^1 q^2 dx + \int_0^1 \psi_t^2 dx \right).$$

Proof. By differentiation of (3.12) and the use of the third and fourth equations of (1.8), we get

$$\begin{aligned} K'_3(t) = & \rho_3 \beta \int_0^1 q \left(\int_0^x \theta(t, y) dy \right) dx + \rho_3 \int_0^1 \theta_x \left(\int_0^x \theta(t, y) dy \right) dx \\ & + \tau \int_0^1 q \left(\int_0^x q_x(t, y) dy \right) dx + \tau \delta \int_0^1 q \left(\int_0^x \psi_{tx}(t, y) dy \right) dx. \end{aligned}$$

By integrating the above equality over $(0, 1)$ and using the boundary conditions $(1.8)_5$ (note also that $\int_0^1 \theta dx = 0$), we have

$$K'_3(t) = \rho_3 \beta \int_0^1 q \left(\int_0^x \theta(t, y) dy \right) dx - \rho_3 \int_0^1 \theta^2 dx + \tau \int_0^1 q^2 dx + \tau \delta \int_0^1 q \psi_t dx.$$

Applying again Young's inequality and the fact that

$$\int_0^1 \left(\int_0^x \theta(t, y) dy \right)^2 dx \leq c \int_0^1 \theta^2 dx,$$

we arrive at (3.13). ■

Lemma 3.5. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.14) \quad K_4(t) := \frac{\tau \rho_2}{k} \int_0^1 \psi_t (\varphi_x + \psi) dx + \frac{b \tau \rho_1}{k^2} \int_0^1 \varphi_t \psi_x dx \\ - \frac{b \tau \rho_3}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 \theta \varphi_t dx + \frac{b \tau}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 q (\varphi_x + \psi) dx,$$

satisfies

$$(3.15) \quad K'_4(t) \leq -(\tau - 2\varepsilon_1) \int_0^1 (\varphi_x + \psi)^2 dx + C \left(\int_0^1 \psi_t^2 dx + \int_0^1 q^2 dx + \int_0^1 h^2(\psi_t) dx \right) \\ + \frac{b \rho_3}{\delta \rho_1} \left[\left(\tau - \frac{\rho_1}{k \rho_3} \right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau \delta^2 \rho_1}{b k \rho_3} \right] \int_0^1 \theta_x (\varphi_x + \psi) dx,$$

with $C = 2 \max(\frac{\tau \rho_2}{k} + \frac{1}{2}, (\frac{b}{\tau k} (\frac{\rho_2}{b} - \frac{\rho_1}{k}))^2 (\frac{\beta^2}{4\varepsilon_1} + \frac{\tau^2}{2}), \frac{c^2 \tau^2}{4k^2 \varepsilon_1})$ and $\varepsilon_1 > 0$.

Proof. By differentiation of (3.14), using (1.8) and integration over $(0, 1)$, we get

$$K'_4(t) = \frac{\tau}{2} \int_0^1 (b \psi_{xx} - k(\varphi_x + \psi) - \delta \theta_x - \alpha(t) h(\psi_t)) (\varphi_x + \psi) dx \\ + \frac{\tau \rho_2}{k} \int_0^1 \psi_t (\varphi_x + \psi)_t dx + \frac{b \tau}{k^2} \int_0^1 (\varphi_x + \psi)_x \varphi_x + \varphi_t \psi_{tx} dx \\ - \frac{b \tau}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 (-(q_x + \delta \psi_{xt}) \varphi_t + \theta (\varphi_x + \psi)_x) dx \\ + \frac{b}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 (-(\beta q + \theta_x) (\varphi_x + \psi) + q (\varphi_x + \psi)_t) dx.$$

By integration over $(0, 1)$ and using the boundary conditions $(1.8)_5$, we have

$$K'_4(t) = -\tau \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\tau \rho_2}{k} \int_0^1 \psi_t^2 dx + \frac{b \tau}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 q \psi_t dx \\ - \frac{b \beta}{\delta k} \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) \int_0^1 q (\varphi_x + \psi) dx - \frac{\tau}{k} \int_0^1 \alpha(t) h(\psi_t) (\varphi_x + \psi) dx \\ + \frac{b \rho_3}{\delta \rho_1} \left[\left(\tau - \frac{\rho_1}{k \rho_3} \right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau \delta^2 \rho_1}{b k \rho_3} \right] \int_0^1 \theta_x (\varphi_x + \psi) dx.$$

Applying Young's inequality, we obtain (3.15). ■

Next, we define a Lyapunov functional K and show that it is equivalent to the energy functional E .

Lemma 3.6. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.16) \quad K(t) := NE(t) + K_1 + N_2 K_2 + N_3 K_3 + N_4 K_4,$$

where N is sufficiently large, N_1 and N_2 are positive real numbers to be chosen properly, satisfies

$$(3.17) \quad c_1 E(t) \leq K(t) \leq c_2 E(t),$$

for c_1 and c_2 two positive constants and

$$(3.18) \quad \begin{aligned} K'(t) \leq & -(\rho_1 - N_2 \rho_1 \varepsilon) \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx - (N_2(b - 2c\varepsilon) - c) \int_0^1 \psi_x^2 dx \\ & - \int_0^1 (N_4(\tau - 2\varepsilon_1) - k)(\varphi_x + \psi)^2 dx - \left(\frac{N_3 \rho_3}{2} - \frac{\delta}{2}\right) \int_0^1 \theta^2 dx \\ & - (N\beta - cN_2 - cN_3 - cN_4) \int_0^1 q^2 dx + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx \\ & + N_4 \frac{b\rho_3}{\delta\rho_1} \left[\left(\tau - \frac{\rho_1}{k\rho_3}\right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\delta^2\rho_1}{bk\rho_3} \right] \int_0^1 \theta_x(\varphi_x + \psi) dx. \end{aligned}$$

Proof. From Lemmas 3.2 to 3.5, we find

$$\begin{aligned} |K(t) - NE(t)| \leq & \rho_1 \int_0^1 |\varphi\varphi_t| dx + (\rho_2 + N_2) \int_0^1 |\psi\psi_t| dx + N_2 \rho_1 \int_0^1 |\varphi_t w| dx \\ & + N_2 \tau \delta \int_0^1 |\psi q| dx + \tau \rho_3 \int_0^1 |q(\int_0^x \theta(t, y) dy)| dx. \end{aligned}$$

Applying Young, Poincaré and Cauchy-Schwartz inequalities and the fact that

$$\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2 \leq 2(\varphi_x + \psi)^2 + 2c\psi_x^2,$$

we obtain (3.17), and therefore we get

$$K(t) \sim E(t).$$

For to prove (3.18), it suffices to differentiate (3.16) and use lemmas 3.1-3.5. This ends the proof of the lemma. \blacksquare

Theorem 3.1. *Let us suppose that*

$$\mu = \left[\left(\tau - \frac{\rho_1}{k\rho_3}\right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\delta^2\rho_1}{bk\rho_3} \right] = 0.$$

Then there exist positive constants k_1, k_2, k_3 and ε_0 such that the energy $E(t)$ associated with (1.8) satisfies

$$(3.19) \quad E(t) \leq k_3 H_1^{-1} \left(k_1 \int_0^t \alpha(s) ds + k_2 \right), \quad \text{for all } t \geq 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds, \quad H_2(t) = tH'(\varepsilon_0 t).$$

Here H_1 is a strictly decreasing and convex function on $(0, 1]$, with $\lim_{t \rightarrow 0} H_1(t) = +\infty$.

Proof. The estimate (3.18), with $\mu = 0$, takes the form

$$\begin{aligned} K'(t) \leq & -(\rho_1 - N_2\rho_1\varepsilon) \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx - (N_2(b - 2c\varepsilon) - c) \int_0^1 \psi_x^2 dx \\ & - \int_0^1 (N_4(\tau - 2\varepsilon_1) - k)(\varphi_x + \psi)^2 dx - \left(\frac{N_3\rho_3}{2} - \frac{\delta}{2}\right) \int_0^1 \theta^2 dx \\ & - (N\beta - cN_2 - cN_3 - cN_4) \int_0^1 q^2 dx + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx. \end{aligned}$$

Now, we choose the constants in the above estimate as follows: first ε and ε_1 are such that

$$\varepsilon = \frac{1}{2N_2} \quad \text{and} \quad \varepsilon_1 < \frac{\tau}{2}.$$

After that, we choose N , N_2 , N_3 and N_4 sufficiently large such that $N_2 > \frac{2c}{b}$, $N_3 > \frac{\delta}{\rho_3}$, $N_4 > \frac{k}{\tau - 2\varepsilon_1}$ and $N > \frac{c}{\beta}(\frac{2c}{b} + \frac{\delta}{\rho_3} + \frac{k}{\tau - 2\varepsilon_1})$. Then, we deduce that

$$(3.20) \quad K'(t) \leq -dE(t) + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx,$$

where $d = \min(\rho_1 - N_2\rho_1\varepsilon, \rho_2, N_2(b - 2c\varepsilon) - c, N_4(\tau - 2\varepsilon_1) - k, \frac{N_3\rho_3}{2} - \frac{\delta}{2}, N\beta - cN_2 - cN_3 - cN_4)$.

First case: Let h_0 be a linear function over $[0, \varepsilon]$. The hypothesis $(A_2)^*$ implies that

$$c'_1|s| \leq |h(s)| \leq c'_2|s|, \quad \text{for all } s \in \mathbb{R}.$$

Consequently, by multiplying inequality (3.20) by $\alpha(t)$, we obtain

$$\begin{aligned} (3.21) \quad \alpha(t)K'(t) & \leq -d\alpha(t)E(t) + c \alpha(t) \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx, \\ & \leq -d\alpha(t)E(t) + c\alpha(t) \int_0^1 \left(\frac{1}{c'_1}|\psi_t h(\psi_t)| + c'_2|\psi_t h(\psi_t)|\right) dx, \\ & \leq -d\alpha(t)E(t) + c_0\alpha(t) \int_0^1 \psi_t h(\psi_t) dx = -d\alpha(t)E(t) - c_0E'(t), \end{aligned}$$

where $c_0 = c(\frac{1}{c'_1} + c'_2)$.

Using now hypothesis (A_1) , this yields

$$(3.22) \quad (\alpha K + c_0 E)'(t) \leq \alpha(t)K'(t) + c_0 E'(t) \leq -d\alpha(t)E(t).$$

We integrate the inequality (3.22) and use the fact that $\alpha K + c_0 E \sim E$, we obtain for some k , $c > 0$,

$$(3.23) \quad E(t) \leq k \exp(-dc \int_0^t \alpha(s) ds).$$

Finally, by a simple computation we get (3.19).

Second case: Let h_0 be a non-linear function over $[0, \varepsilon]$. We assume that $\max(r, h_0(r)) < \varepsilon$, where r is defined in the hypothesis $(A_2)^*$.

Let $\varepsilon_1 = \min(r, h_0(r))$, we deduce from the hypothesis $(A_2)^*$ that

$$\frac{h_0(\varepsilon_1)}{\varepsilon}|s| \leq \frac{h_0(|s|)}{|s|}|s| \leq |h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0(\varepsilon)}{\varepsilon_1}|s|,$$

for all s satisfying $\varepsilon_1 \leq |s| \leq \varepsilon$.

Then, the estimates in hypothesis $(A_2)^*$ become

$$(3.24) \quad \begin{cases} h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|), & \text{for all } |s| \leq \varepsilon_1, \\ c'_1|s| \leq |h(s)| \leq c'_2|s|, & \text{for all } |s| \geq \varepsilon_1, \end{cases}$$

and we have

$$(3.25) \quad s^2 + h^2(s) \leq 2H^{-1}(sh(s)).$$

To estimate the last term of (3.20), we consider the following partition of (0.1):

$$\Omega_1 = \{x \in (0, 1); |\psi_t| \leq \varepsilon_1\}, \quad \Omega_2 = \{x \in (0, 1); |\psi_t| > \varepsilon_1\}.$$

Then, we obtain

$$(3.26) \quad \psi_t h(\psi_t) \leq H(r^2) \quad \text{and} \quad \psi_t h(\psi_t) \leq r^2 \quad \text{on } \Omega_1.$$

Now, we apply Jensen's inequality to the following term

$$I(t) := \frac{1}{|\Omega_1|} \int_{\Omega_1} \psi_t h(\psi_t) dx,$$

and we infer that

$$(3.27) \quad H^{-1}(I(t)) \geq c \int_{\Omega_1} H^{-1}(\psi_t h(\psi_t)) dx.$$

Using (3.24), (3.25) and (3.27), then the right-hand side of (3.20) multiplied by $\alpha(t)$ becomes

$$\begin{aligned} \alpha(t) \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx &= \alpha(t) \int_{\Omega_1} (\psi_t^2 + h^2(\psi_t)) dx + \alpha(t) \int_{\Omega_2} (\psi_t^2 + h^2(\psi_t)) dx, \\ &\leq 2\alpha(t) \int_{\Omega_1} H^{-1}(\psi_t h(\psi_t)) dx \\ &\quad + \alpha(t) \int_{\Omega_2} (|\psi_t| \frac{1}{c'_1} |h(\psi_t)| + c'_2 |\psi_t| |h(\psi_t)|) dx, \\ &\leq c\alpha(t) H^{-1}(I(t)) + \alpha(t) c \int_0^1 \psi_t h(\psi_t) dx, \\ &\leq c\alpha(t) H^{-1}(I(t)) - cE'(t). \end{aligned}$$

Consequently, the estimate (3.20) gives

$$(3.28) \quad R'_0(t) \leq -d\alpha(t)E(t) + c\alpha(t)H^{-1}(I(t)),$$

where $R_0 = \alpha K + cE$.

On the one hand, for $\varepsilon_0 < r^2$, using (3.28), $H' \geq 0$ and $H'' \geq 0$ over $(0, r^2]$ and $E' \leq 0$ the functional R_1 defined by

$$R_1(t) := H' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) R_0(t) + c_0 E(t),$$

is equivalent to $E(t)$.

On the other hand, using the fact that $\varepsilon_0 \frac{E'(t)}{E(0)} H''(\varepsilon_0 \frac{E(t)}{E(0)}) R_0(t) \leq 0$ and (3.28),

we conclude that

$$(3.29) \quad \begin{aligned} R_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H''(\varepsilon_0 \frac{E(t)}{E(0)}) R_0(t) + H'(\varepsilon_0 \frac{E(t)}{E(0)}) R_0'(t) + c_0 E'(t) \\ &\leq -d\alpha(t) E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\alpha(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) H^{-1}(I(t)) + c_0 E'(t). \end{aligned}$$

Our goal now is to estimate the second term in the right-hand side of (3.29). For that purpose, we introduce the convex conjugate H^* of H defined by

$$(3.30) \quad H^*(s) = s(H')^{-1}(s) - H((H')^{-1}(s)) \text{ for } s \in (0, H'(r^2)),$$

and H^* satisfies the following Young inequality:

$$(3.31) \quad AB \leq H^*(A) + H(B) \text{ for } A \in (0, H'(r^2)), B \in (0, r^2).$$

Now, taking $A = H'(\varepsilon_0 \frac{E(t)}{E(0)})$ and $B = H^{-1}(I(t))$, we obtain

$$\begin{aligned} R_1'(t) &\leq -d\alpha(t) E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\alpha(t) H^*\left(H'(\varepsilon_0 \frac{E(t)}{E(0)})\right) \\ &\quad + c\alpha(t) H(H^{-1}(I(t))) + c_0 E'(t) \\ &\leq -d\alpha(t) E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\varepsilon_0 \frac{E(t)}{E(0)} \alpha(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) \\ &\quad - c\alpha(t) H(\varepsilon_0 \frac{E(t)}{E(0)}) + c\alpha(t) I(t) + c_0 E'(t) \\ &\leq -d\alpha(t) E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\varepsilon_0 \frac{E(t)}{E(0)} \alpha(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) - cE'(t) + c_0 E'(t). \end{aligned}$$

With a suitable choice of ε_0 and c_0 , we deduce from the last inequality that

$$(3.32) \quad R_1'(t) \leq -(dE(0) - c\varepsilon_0) \alpha(t) \frac{E(t)}{E(0)} H'(\varepsilon_0 \frac{E(t)}{E(0)}) \leq -k\alpha(t) H_2(\frac{E(t)}{E(0)}),$$

where $k = dE(0) - c\varepsilon_0 > 0$ and $H_2(s) = sH'(\varepsilon_0 s)$.

Since $E(t) \sim R_1(t)$, then there exist a_1 and a_2 such that

$$a_1 R_1(t) \leq E(t) \leq a_2 R_1(t).$$

We set now $R(t) = \frac{a_1 R_1(t)}{E(0)}$. It is clear that $R(t) \sim E(t)$. We use the fact that $H_2'(t)$, $H_2(t) > 0$ over $(0, 1]$ (this is due to the fact that H is strictly convex on $(0, r^2]$) and we deduce from (3.44) that

$$R'(t) \leq -k_1 \alpha(t) H_2(R(t)), \quad \text{for all } t \in \mathbb{R}_+,$$

with $k_1 > 0$.

By integrating the last inequality, we obtain

$$H_1(R(t)) \geq H_1(R(0)) + k_1 \int_0^t \alpha(s) ds.$$

Finally, using the fact that H_1^{-1} is decreasing (because H_1 is also), we have

$$R(t) \leq H_1^{-1}\left(k_1 \int_0^t \alpha(s) ds + k_2\right), \quad \text{with } k_2 > 0.$$

Taking into account that $E(t) \sim R(t)$, we deduce (3.19).



3.1.1. *Examples.* In the following, we will apply the inequality (3.19) on some examples in order to show explicit stability results in term of asymptotic profiles in time. For that, we choose the function H strictly convex near zero.

Example 1.

Let h be a function that satisfies

$$c_3 \min(|s|, |s|^p) \leq |h(s)| \leq c_4 \max(|s|, |s|^{\frac{1}{p}}),$$

with some $c_3, c_4 > 0$ and $p \geq 1$.

For $h_0(s) = cs^p$, hypothesis $(A_2)^*$ is verified. Then $H(s) = cs^{\frac{p+1}{2}}$.

Therefore, we distinguish the following two cases:

- If $p=1$, we have h_0 is linear and $H_2(s) = cs$, $H_1(s) = -\frac{\ln(s)}{c}$ and $H_1^{-1}(t) = \exp(-ct)$.

Applying (3.19) of Theorem 3.1, we conclude that

$$E(t) \leq k_3 \exp(-c(k_1 \int_0^t \alpha(s) ds + k_2)).$$

- If $p > 1$; this implies that h_0 is nonlinear and we have $H_2(s) = c^{\frac{p+1}{2}} \varepsilon_0^{\frac{p-1}{2}} s^{\frac{p-1}{2}}$ and

$$H_1(t) = \int_t^1 \frac{1}{\delta} s^{-\frac{p-1}{2}} ds = \frac{2}{\delta(1-p)} - \frac{2}{\delta(1-p)} t^{\frac{p-1}{2}}, \quad \text{with } \delta = c^{\frac{p+1}{2}} \varepsilon_0^{\frac{p-1}{2}}.$$

Therefore,

$$H_1^{-1}(t) = (\delta^{\frac{p-1}{2}} t + 1)^{-\frac{2}{p-1}}.$$

Using again (3.19), we obtain

$$E(t) \leq H_1^{-1}(k_1 \int_0^t \alpha(s) ds + k_2) = (\delta^{\frac{p-1}{2}} (k_1 \int_0^t \alpha(s) ds + k_2) + 1)^{-\frac{2}{p-1}}.$$

Example 2.

Let $h_0(s) = \exp(-\frac{1}{s})$, this yields $H(s) = \sqrt{s} \exp(-\frac{1}{\sqrt{s}})$ and

$$H_2(s) = (\frac{\sqrt{s}}{2\sqrt{\varepsilon_0}} + \frac{1}{2\varepsilon_0}) \exp(-\frac{1}{\sqrt{\varepsilon_0 s}}).$$

Moreover, we have

$$\begin{aligned} H_1(t) &= \int_t^1 \left(\frac{1}{\frac{\sqrt{s}}{2\sqrt{\varepsilon_0}} + \frac{1}{2\varepsilon_0}} \right) \exp(-\frac{1}{\sqrt{\varepsilon_0 s}}) ds \\ &\leq \int_t^1 \frac{2\sqrt{\varepsilon_0}}{\sqrt{s}} \exp(-\frac{1}{\sqrt{\varepsilon_0 s}}) ds \\ &\leq c \int_t^1 \frac{1}{2s\sqrt{\varepsilon_0 s}} \exp(-\frac{1}{\sqrt{\varepsilon_0 s}}) ds = c \exp(-\frac{1}{\sqrt{\varepsilon_0 t}}) - c \exp(-\frac{1}{\sqrt{\varepsilon_0}}). \end{aligned}$$

Then,

$$t \leq \varepsilon_0^{-1} \left(\ln \left(\frac{H_1(t) + c \exp(-\frac{1}{\sqrt{\varepsilon_0 s}})}{c} \right) \right)^{-2}.$$

Replacing t by $H_1^{-1} \left(k_1 \int_0^t \alpha(s) ds + k_2 \right)$ in the last inequality, we find

$$H_1^{-1} \left(k_1 \int_0^t \alpha(s) ds + k_2 \right) \leq \varepsilon_0^{-1} \left(\ln \left(\frac{k_1 \int_0^t \alpha(s) ds + k_2 + c \exp(\frac{1}{\sqrt{\varepsilon_0}})}{c} \right) \right)^{-2}.$$

Therefore,

$$E(t) \leq k_3 \varepsilon_0^{-1} \left(\ln \left(\frac{k_1 \int_0^t \alpha(s) ds + k_2 + c \exp(\frac{1}{\sqrt{\varepsilon_0}})}{c} \right) \right)^{-2}.$$

Example 3.

Let $h_0(s) = \frac{1}{s} \exp(-\frac{1}{s^2})$. Following the same steps in exemple 2 we find that the energy of (1.8) satisfies

$$E(t) \leq \varepsilon \left(\ln \left(\frac{k_1 \int_0^t \alpha(s) ds + k_2 + c \exp(\frac{1}{\varepsilon_0})}{c} \right) \right)^{-1}.$$

Example 4.

Let $h_0(s) = \frac{1}{s} \exp(-\frac{1}{4}(\ln s)^2)$. Then, we have $H(s) = \exp(-\frac{1}{4}(\ln s)^2)$, $H_2(s) = -\frac{1}{2} \frac{\ln \varepsilon_0 s}{\varepsilon_0} \exp(-\frac{1}{4}(\ln \varepsilon_0 s)^2)$ and $H_1(t) = \int_t^1 -2 \frac{\varepsilon_0}{\ln \varepsilon_0 s} \exp(\frac{1}{4}(\ln \varepsilon_0 s)^2) ds$.

As $\lim_{s \rightarrow 0} \frac{4\varepsilon_0^2 s}{(\ln(\varepsilon_0 s))^2} = 0$, then the function $s \mapsto \frac{4\varepsilon_0^2 s}{(\ln(\varepsilon_0 s))^2}$ is bounded on $(0, 1]$, and we infer that

$$H_1(t) \leq c \int_t^1 -\frac{1}{2} \frac{\ln \varepsilon_0 s}{\varepsilon_0 s} \exp(\frac{1}{4}(\ln s)^2) ds = \exp(\frac{1}{4}(\ln \varepsilon_0 t)^2) - \underbrace{\exp(\frac{1}{4}(\ln \varepsilon_0)^2)}_{c_1}.$$

Hence, we have

$$t \leq \frac{1}{\varepsilon_0} \exp \left(-2 (\ln(H_1(t)) + c_1)^{\frac{1}{2}} \right).$$

Replacing t by $H_1^{-1} \left(k_1 \int_0^t \alpha(s) ds + k_2 \right)$ in the last inequality, we find

$$E(t) \leq k_3 H_1^{-1} \left(k_1 \int_0^t \alpha(s) ds + k_2 \right) = \frac{k_3}{\varepsilon_0} \exp \left(-2 \left(\ln k_1 \int_0^t \alpha(s) ds + k_2 + c_1 \right)^{\frac{1}{2}} \right).$$

3.2. The case $\mu \neq 0$ and $\alpha(t) = 1$.

This section is devoted to the statement and the proof of the stability result for the system (1.8) when $\mu \neq 0$ and $\alpha(t) = 1$.

We have the following theorem.

Theorem 3.2. *Let us suppose that conditions (A_1) and $(A_2)^*$ hold, then for*

$$\mu = \left[\left(\tau - \frac{\rho_1}{k\rho_3} \right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau\delta^2\rho_1}{bk\rho_3} \right] \neq 0,$$

the energy solution of (1.8) satisfies

$$(3.33) \quad E(t) \leq H_2^{-1}\left(\frac{c}{t}\right),$$

where

$$H_2(t) = tH'(\varepsilon_0 t) \text{ with } \lim_{t \rightarrow 0} H_2(t) = 0.$$

Proof. Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). First, we define

$$(3.34) \quad E(t) := \frac{1}{2} \int_0^1 (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b \psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau q^2) dx,$$

and

$$(3.35) \quad \tilde{E}(t) := \frac{1}{2} \int_0^1 (\rho_1 \varphi_{tt}^2 + \rho_2 \psi_{tt}^2 + b \psi_{tx}^2 + k(\varphi_{tx} + \psi_t)^2 + \rho_3 \theta_t^2 + \tau q_t^2) dx.$$

Then, the functional E satisfies

$$(3.36) \quad E'(t) = -\beta \int_0^1 q^2 dx - \int_0^1 \psi_t h(\psi_t) dx \leq 0.$$

Analogously, the functional \tilde{E} satisfies

$$(3.37) \quad \tilde{E}'(t) = -\beta \int_0^1 q_t^2 dx - \int_0^1 \psi_{tt}^2 h'(\psi_t) dx \leq 0.$$

Using the results in Subsection 3.1 (recall the expressions of the functionals K_1, \dots, K_4) we have the following Lemma.

Lemma 3.7. *Let $(\varphi, \psi, \theta, q)$ be a solution of the system (1.8). Then, the functional*

$$(3.38) \quad L(t) := N(E(t) + \tilde{E}(t)) + K_1 + N_2 K_2 + N_3 K_3 + N_4 K_4,$$

satisfies

$$(3.39) \quad L'(t) \leq -d'(t) + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx,$$

for N large enough and $d' > 0$.

Proof. By differentiation of (3.38), and using (3.18) and Young's inequality, we obtain

$$(3.40) \quad \begin{aligned} L'(t) \leq & -dE(t) + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx + c \int_0^1 (\theta_x^2 + (\varphi_x + \psi)^2) dx \\ & - N\beta \int_0^1 q_t^2 dx - N \int_0^1 \psi_{tt}^2 h'(\psi_t) dx. \end{aligned}$$

Now, from (1.8)₄, we deduce that

$$\int_0^1 \theta_x^2 dx \leq c \left(\int_0^1 q^2 dx + \int_0^1 q_t^2 dx \right).$$

Consequently, we get

$$(3.41) \quad \begin{aligned} L'(t) \leq & -d'E(t) + c \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx - (\beta N - c) \int_0^1 q_t^2 dx \\ & - N \int_0^1 \psi_{tt}^2 h'(\psi_t) dx. \end{aligned}$$

where $d' = d - c > 0$ and d is the same constant that appears in (3.20). Finally, we choose N large enough and using the monotonicity of the function h we arrive at (3.39). \blacksquare

Now, using the following partition of $(0, 1)$ defined in Subsection 3.1, the right-hand side of (3.39) becomes

$$\int_0^1 (\psi_t^2 + h^2(\psi_t)) dx = \int_{\Omega_1} (\psi_t^2 + h^2(\psi_t)) dx + \int_{\Omega_2} (\psi_t^2 + h^2(\psi_t)) dx.$$

Now, the estimates (3.24)-(3.27) imply that

$$\begin{aligned} \int_0^1 (\psi_t^2 + h^2(\psi_t)) dx &\leq 2 \int_{\Omega_1} H^{-1}(\psi_t h(\psi_t)) dx \\ &\quad + \int_{\Omega_2} (|\psi_t| \frac{1}{c_1} |h(\psi_t)| + c_2' |\psi_t| |h(\psi_t)|) dx \\ &\leq cH^{-1}(I(t)) + c \int_0^1 \psi_t h(\psi_t) dx. \end{aligned}$$

Consequently,

$$\begin{aligned} L'(t) &\leq -d'E(t) + cH^{-1}(I(t)) + c \int_0^1 \psi_t h(\psi_t) dx + c\beta \int_0^1 q^2 dx \\ &\leq -d'E(t) + cH^{-1}(I(t)) - cE'(t). \end{aligned}$$

Hence, we deduce that

$$(3.42) \quad (L + cE)'(t) \leq -d'E(t) + cH^{-1}(I(t)).$$

We then define

$$R_1(t) := H'(\varepsilon_0 \frac{E(t)}{E(0)})(L + cE)(t) + c_0 E(t),$$

which verifies

$$(3.43) \quad R_1'(t) \leq -d_1 E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + cH'(\varepsilon_0 \frac{E(t)}{E(0)}) H^{-1}(I(t)) + \epsilon E'(t),$$

as we have $\varepsilon_0 \frac{E'(t)}{E(0)} H''(\varepsilon_0 \frac{E(t)}{E(0)}) R_0(t) \leq 0$.

We recall the definition of the convex conjugate H^* of H , given by (3.30), which satisfies the following Young inequality:

$$AB \leq H^*(A) + H(B) \text{ for } A \in (0, H'(r^2)), B \in (0, r^2).$$

With the same choice of A and B as in (3.31), we obtain

$$R_1'(t) \leq -d_1 E(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\varepsilon_0 \frac{E(t)}{E(0)} \alpha(t) H'(\varepsilon_0 \frac{E(t)}{E(0)}) - cE'(t) + \epsilon E'(t).$$

With a suitable choice of ε_0 and ϵ , we deduce from the above inequality that

$$(3.44) \quad R_1'(t) \leq -(dE(0) - c\varepsilon_0) \frac{E(t)}{E(0)} H'(\varepsilon_0 \frac{E(t)}{E(0)}) \leq -k\alpha(t) H_2(\frac{E(t)}{E(0)}),$$

where $k = dE(0) - c\varepsilon_0 > 0$ and $H_2(s) = sH'(\varepsilon_0(s))$.

Finally, we have

$$R_1'(t) \leq -k_1 H_2\left(\frac{E(t)}{E(0)}\right), \quad \text{for all } t \in \mathbb{R}_+,$$

with $k_1 > 0$, which yields

$$t H_2\left(\frac{E(t)}{E(0)}\right) \leq \int_0^t H_2\left(\frac{E(s)}{E(0)}\right) ds \leq -(R_1(t) - R_1(0)) \leq R_1(0).$$

Then, we easily deduce that

$$H_2\left(\frac{E(t)}{E(0)}\right) \leq \frac{R_1(0)}{t}.$$

Thus,

$$E(t) \leq E(0) H_2^{-1}\left(\frac{R_1(0)}{t}\right).$$

This concludes the proof of Theorem 3.2. ■

3.2.1. Examples.

Example 1: Let $h_0(s) = cs^p$. Then $H(s) = cs^{\frac{p+1}{2}}$.

Therefore, we distinguish the following two cases:

- If $p=1$, we have h_0 is linear and $H_2^{-1}(t) = cs$.

Applying (3.33) of Theorem 3.2, we conclude that

$$E(t) \leq \frac{c}{t}.$$

- If $p > 1$; this implies that h_0 is nonlinear and we have $H_2(s) = cs^{\frac{p-1}{2}}$. Therefore,

$$H_2^{-1}(t) = ct^{\frac{2}{p-1}}.$$

Using (3.33), we obtain

$$E(t) \leq ct^{-\frac{2}{p-1}}.$$

Examples 2: Let h be given by $h(x) = \frac{1}{x^3} \exp(-\frac{1}{x^2})$ and we choose $h_0(x) = \frac{1+x^2}{x^3} \exp(-\frac{1}{x^2})$,

we obtain $H(x) = \frac{1+x}{x} \exp(-\frac{1}{x})$ and $H_2(x) = \frac{\exp(-\frac{1}{\varepsilon_0 x})}{\varepsilon_0^3 x^2}$.

Then, we use the following property :

$$\lim_{x \rightarrow 0^+} \exp\left(\frac{1}{\varepsilon_0 x}\right) H_2(x) = +\infty,$$

and we deduce that

$$\exp\left(-\frac{1}{\varepsilon_0 x}\right) \leq H_2(x).$$

We infer that there exists $x_0 > 0$ such that,

$$\exp\left(-\frac{1}{\varepsilon_0 x}\right) \leq H_2(x) \text{ on } (0, x_0].$$

Consequently, the energy of the solution of (1.8) satisfies the estimate

$$E(t) \leq c(\ln(t))^{-1}.$$

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